Chapter five

Multiple Integrals

In this chapter we consider the integral of a function of two variables $f(x,y)$ over a region in the plane and the integral of a function of three variables $f(x,y,z)$ over a region in space. These integrals are called *multiple integrals*

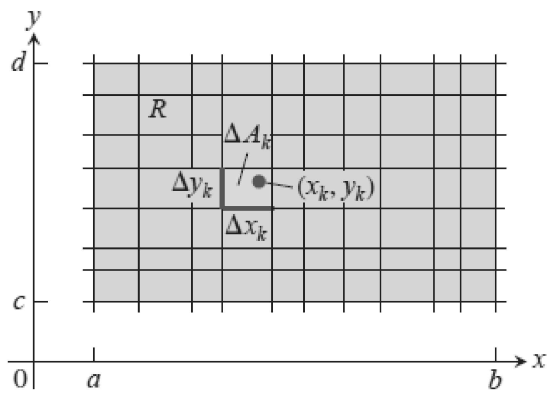
Double integrals:

Double integrals over rectangles:

We begin our investigation of double integrals by considering the simplest type of planar region, a rectangle. We consider a function $f(x,y)$ defined on a rectangular region R ,

$$R: \ a \le x \le b \ , \ c \le y \le d$$

We subdivide R into small rectangles using a network of lines parallel to the x and y- axes. The lines divide R into n rectangular pieces, where the number of such pieces n gets large as the width and height of each piece gets small.



These rectangles form a **partition** of R . a small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$

If we number the small pieces partitioning R in some order, then their areas are given by number $\Delta A_1, \Delta A_2$, …. , ΔA_n

Where ΔA_k is the area of the k_{th} small rectangle.

To from a Riemann sum over R , we choose a point (x_k, y_k) in the k_{th} small rectangle, multiply the value of f at the point by the area ΔA_k , and added together the products:

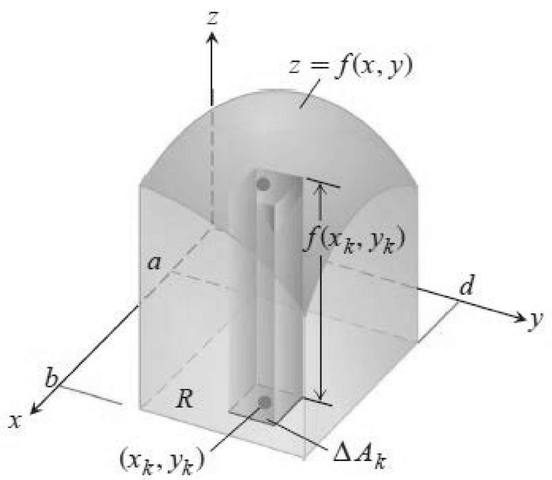
$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be **integral** and the limit is called the **double integral** of f over R , written as

$$\iint_R f(x,y)dA \qquad or \qquad \iint_R f(x,y)dxdy$$

Double integrals as volume:
When $f(x,y)$ is a positive function over a rectangular region R in the xy – plane, we may interpret the double integral of f over R as the volume of the 3- dimensional solid region over xy – plane bounded below by R and above by the surface $z = f(x,y)$



Each term $f(x_k, y_k)\Delta A_k$ in the sum $S_n = \sum f(x_k, y_k)\Delta A_k$ is the volume of a vertical rectangular box that approximates the volume of the portion of the solid that stands directly above the base ΔA_k. The sum S_n thus approximates what we want to call the total volume of the solid. We define this volume to be

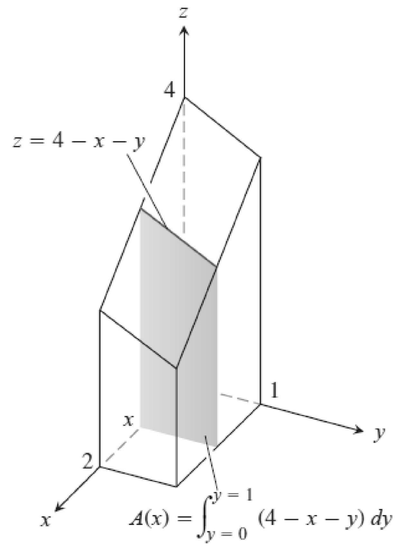
$$volume = \lim_{n\to\infty} S_n = \iint_R f(x,y)dA$$

Where $\Delta A_k \to 0$ as $n \to \infty$

Fubini's theorem for calculating double integrals:

Suppose that we wish to calculate the volume under the plane $z = 4 - x - y$ over the rectangular region R: $0 \le x \le 2$, $0 \le y \le 1$

$$\int_{x=0}^{x=2} A(x)dx$$



Where $A(x)$ is the cross – sectional area at x

For each value of x , we may calculate $A(x)$ as the Integral:

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)dy$$

$$volume = \int_{x=0}^{x=2} A(x)dx = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y)dy \right) dx$$

$$= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx = \int_{x=0}^{x=2} \left[(4)(1) - (x)(1) - \frac{(1)^2}{2} - 0 \right] dx$$

$$= \int_{x=0}^{x=2} \left(4 - x - \frac{1}{2}\right) dx = \int_{x=0}^{x=2} \left(\frac{7}{2} - x\right) dx$$

$$= \left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2 = \left[\left(\frac{7}{2}\right)(2) - \frac{(2)^2}{2} - 0\right] = \frac{14}{2} - \frac{4}{2} = \frac{10}{2} = 5$$

If we just wanted to write a formula for the volume, without carrying out any of the integrations, we could write

$$volume = \int_0^2 \int_0^1 (4 - x - y)\,dy\,dx$$

The expression on the right, called an **iterated** or **repeated integral**

Fubini's theorem (first form)

If $f(x, y)$ is continuous throughout the rectangular region

R: $a \le x \le b$, $c \le y \le d$

Then

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

Example: Calculate

$$\iint_R f(x,y)dA$$

for $f(x,y)=1-6x^2y$ and R: $0\le x\le 2$, $-1\le y\le 1$

Solution: by fubini's theorem:

$$\iint_R f(x,y)dA=\int_{-1}^{1}\int_{0}^{2}(1-6x^2y)dxdy=\int_{-1}^{1}\left[x-\frac{6x^3y}{3}\right]_{x=0}^{x=2}dy$$

$$=\int_{-1}^{1}[x-2x^3y]_{x=0}^{x=2}dy=\int_{-1}^{1}(2-(2)(2)^3y)dy$$

$$=\int_{-1}^{1}[2-16y]dy$$

$$=\left[2y-\frac{16y^3}{2}\right]_{-1}^{1}=[2y-8y^2]_{-1}^{1}$$

$$=\left((2)(1)-(8)(1)^2\right)-\left((2)(-1)-(8)(-1)^2\right)$$

$$=(2-8)-(-2-8)$$

$$=(-6)-(-10)$$

$$=-6+10=4$$

Reversing the order of integration gives the same answer

$$=\int_{0}^{2}\int_{-1}^{1}(1-6x^2y)dydx=\int_{0}^{2}\left[y-\frac{6x^2y^2}{2}\right]_{y=-1}^{y=1}dx=\int_{0}^{2}[y-3x^2y^2]_{y=-1}^{y=1}dx$$

$$=\int_{0}^{2}[(1-3x^2)-(-1-3x^2)]\,dx$$

$$= \int_0^2 (1 - 3x^2 + 1 + 3x^2)\, dx$$

$$= \int_0^2 2\, dx = [2x]_0^2$$

$$= (2)(2) - 0 = 4$$

***Example*:** Evaluate the integral

$$\int_0^3 \int_1^2 (1 + 8xy)dydx$$

***Solution*:**

$$= \int_0^3 \int_1^2 (1 + 8xy)dydx = \int_0^3 \left[y + \frac{8xy^2}{2}\right]_{y=1}^{y=2} dx$$

$$= \int_0^3 [y + 4xy^2]_{y=1}^{y=2}\, dx$$

$$= \int_0^3 [(2 + (4)(x)(2)^2) - (1 + (4)(x)(1)^2)]\, dx$$

$$= \int_0^3 [(2 + 16x) - (1 + 4x)]\, dx = \int_0^3 (2 + 16x - 1 - 4x)dx$$

$$= \int_0^3 (1 + 12x)dx = \left[x + \frac{12x^2}{2}\right]_0^3$$

$$= [x + 6x^2]_0^3 = (3 + (6)(3)^2 - 0)$$

$$= 3 + 54 = 57$$

Example: Evaluate the integral

$$\int_0^1 \int_0^2 (x+3)dydx$$

Solution:

$$\int_0^1 \int_0^2 (x+3)dydx = \int_0^1 [xy+3y]_{y=0}^{y=2} dx$$

$$= \int_0^1 (2x+(3)(2)-0)dx$$

$$= \int_0^1 (2x+6)dx$$

$$= \left[\frac{2x^2}{2}+6x\right]_0^1 = [x^2+6x]_0^1$$

$$= [1+(6)(1)-0]$$

$$= 1+6 = 7$$

Example: Evaluate the integral

$$\int_0^3 \int_0^2 (4-y^2)dydx$$

Solution:

$$\int_0^3 \int_0^2 (4-y^2)dydx = \int_0^3 \left[4y - \frac{y^3}{3}\right]_{y=0}^{y=2} dx$$

$$= \int_0^3 \left((4)(2) - \frac{(2)^3}{3}\right) dx$$

$$= \int_0^3 \left(8 - \frac{8}{3}\right) dx = \int_0^3 \left(\frac{24-8}{3}\right) dx$$

$$= \int_0^3 \left(\frac{16}{3}\right) dx = \frac{16}{3}\int_0^3 dx$$

$$= \frac{16}{3}[x]_0^3 = \left(\frac{16}{3}\right)(3) = 16$$

H.W: Evaluate the integral

1. $\int_0^3 \int_{-2}^0 (x^2y - 2xy)dydx$

2. $\int_{-1}^0 \int_{-1}^1 (x+y+1)dxdy$

3. $\int_1^3 \int_{-1}^1 (2x-4y)dydx$

4. $\int_2^4 \int_0^1 x^2ydxdy$

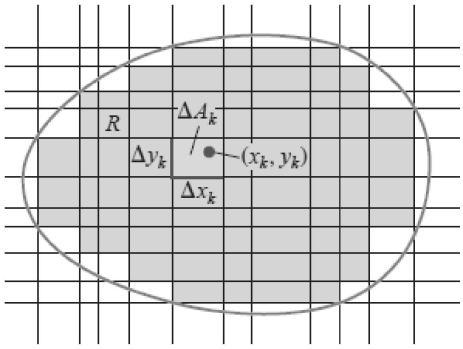
5. $\int_{\pi}^{2\pi} \int_0^{\pi} (sinx + cosy)dxdy$

Double integrals over bounded nonrectangular regions:

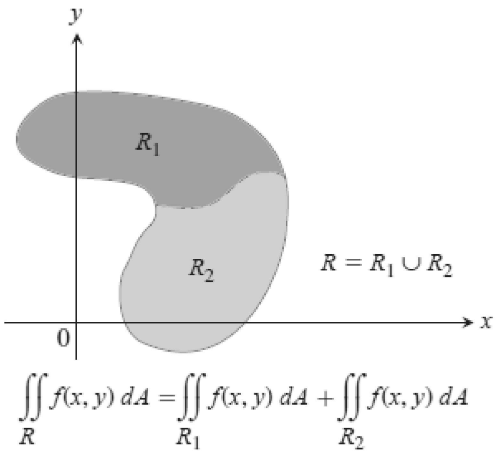
To define the double integral of a function $f(x,y)$ over a bounded, nonrectangular region R ,

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

$$\lim_{\Delta A \to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \iint_R f(x,y)\,dA$$



$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$



Fubini's theorem (stronger form):

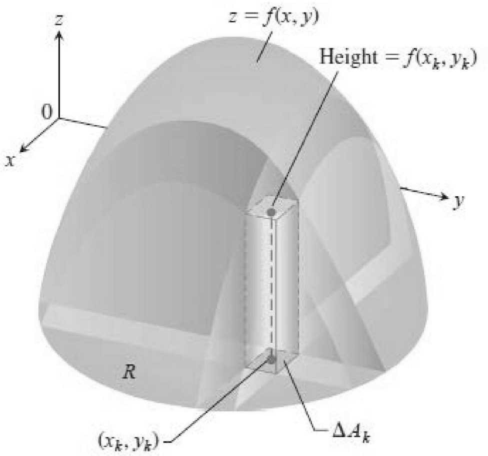
Let $f(x,y)$ be continuous on a region R

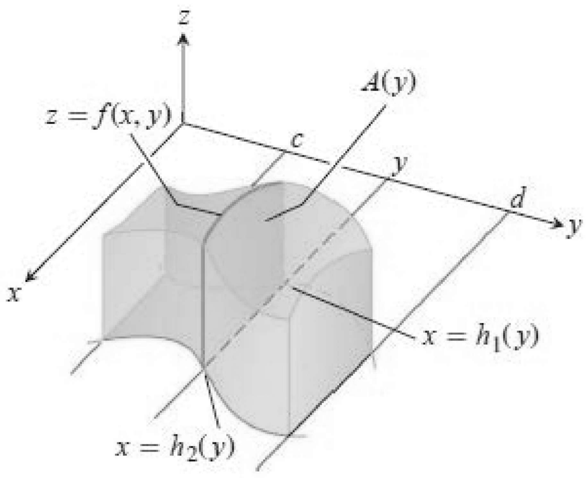
1. If R is define by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then:

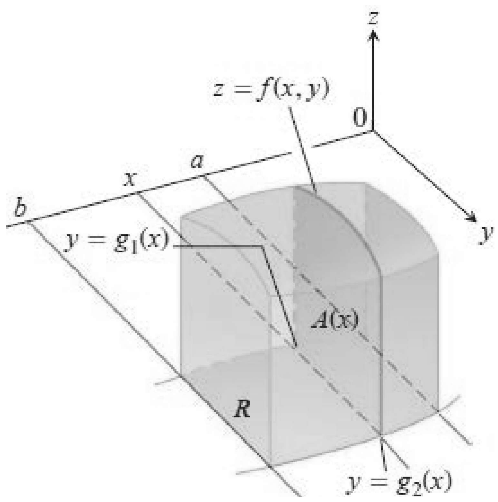
$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

2. If R is define by $c \le y \le d$, $h_1(y) \le y \le h_2(y)$, with h_1 and h_2 continuous on $[a,b]$, then:

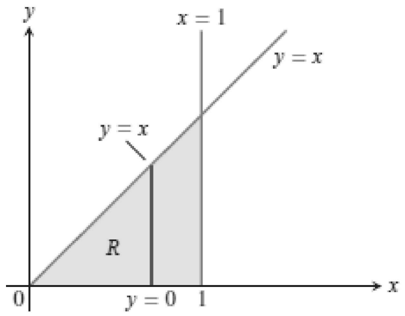
$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

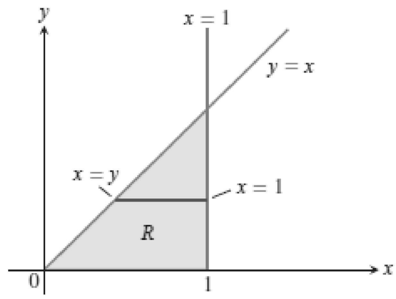


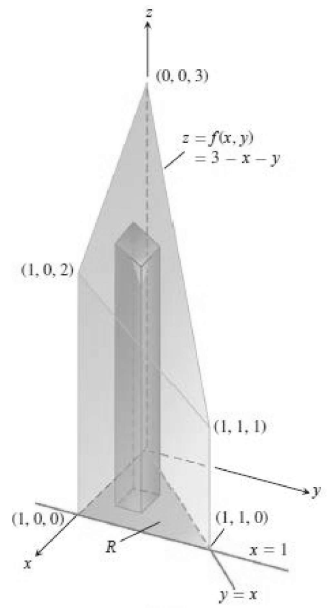




Example: Find the volume of the prism whose is the triangle in the $xy-$ plane bounded by the $x-$ axis and the lines $y=x$ and $x=1$ and whose top lies in the plane $z=f(x,y)=3-x-y$







Solution: from the figure , for any x between 0 and 1, y may vary from $y=0$ to $y=x$, hence

$$v=\int_0^1\int_0^x(3-x-y)dydx=\int_0^1\left[3y-xy-\frac{y^2}{2}\right]_{y=0}^{y=x}dx$$

$$=\int_0^1(3x-x^2-\frac{x^2}{2})\,dx=\int_0^1(3x-\frac{3x^2}{2})dx$$

$$=\left[\frac{3x^2}{2}-\frac{3x^3}{6}\right]_{x=0}^{x=1}=\left[\frac{3x^2}{2}-\frac{x^3}{2}\right]_0^1$$

$$=\left(\frac{3}{2}-\frac{1}{2}\right)=\frac{2}{2}=1$$

When the order of integration is reversed the integral for the volume is:

$$v = \int_0^1 \int_y^1 (3 - x - y)dxdy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left[\left(3 - \frac{1}{2} - y\right) - \left(3y - \frac{y^2}{2} - y^2\right)\right] dy$$

$$= \int_0^1 (3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2)\, dy$$

$$= \int_0^1 (\frac{5}{2} - 4y + \frac{3}{2}y^2)\, dy$$

$$= \left[\frac{5}{2}y - \frac{4y^2}{2} + \frac{3y^3}{6}\right]_{y=0}^{y=1} = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_0^1$$

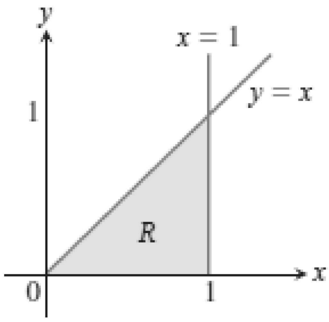
$$= \frac{5}{2} - 2 + \frac{1}{2}$$

$$= \frac{6}{2} - 2 = 3 - 2 = 1$$

***Example*:** Calculate

$$\iint_R \frac{sinx}{x} dA$$

Where R is the triangle in the xy – plane bounded by x – axis, the line $y = x$ and the line $x = 1$



***Solution*:**

$$\int_0^1 \int_0^x \frac{sinx}{x} dydx = \int_0^1 \left[y \frac{sinx}{x}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(\frac{xsinx}{x}\right) dx = \int_0^1 sinxdx$$

$$= [-cosx]_0^1 = [-\cos(1) - (-cos0)]$$

$$= -\cos(1) + 1 \cong 046$$

If we reverse the order of integration and attempt to calculate:

$$\int_0^1 \int_y^1 \frac{sinx}{x} dxdy$$

H.W: Evaluate the integral

1. $\int_0^1 \int_y^1 x^2 e^{xy} dxdy$ **2.** $\int_1^2 \int_y^{y^2} dxdy$

Finding limits of integration:

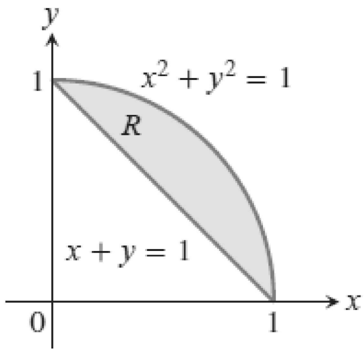
We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with evaluating

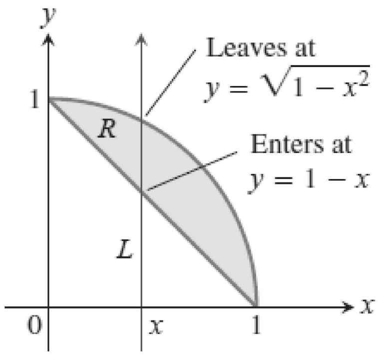
$$\iint_R f(x,y)dA$$

integration first with respect to *y* and then with respect to *x* , do the following:

1. *Sketch*. Sketch the region of integration and label the bounding curves.

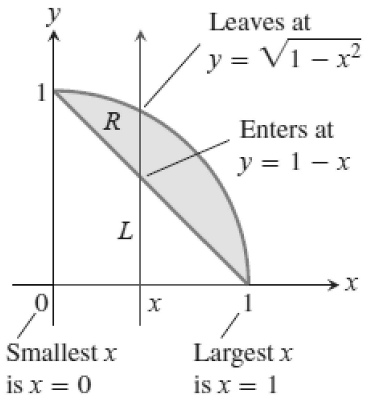


2. *Find the y – limits of integration*. Imagine a vertical line *L* cutting through R in the direction of increasing y. mark the *y* – value where *L* enters and leaves. These are the *y* – limits of integration and are usually functions of *x* (instead of constants)



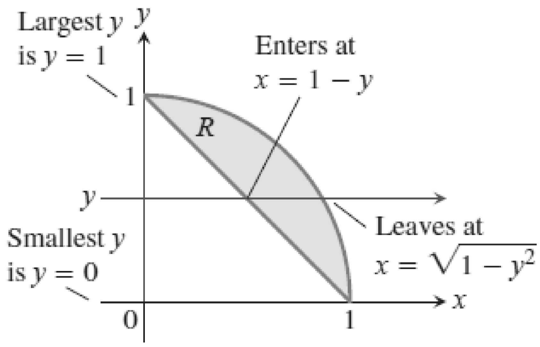
3. Find the x – limits of integration. Choose x – limits that include all the vertical lines through R. the integral shown here is:

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx$$



To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical line in step **2** and **3**. The integral is

$$\iint_R f(x,y)\,dA = \int_{0}^{1}\int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$



Example: Sketch the region of integration for the integral

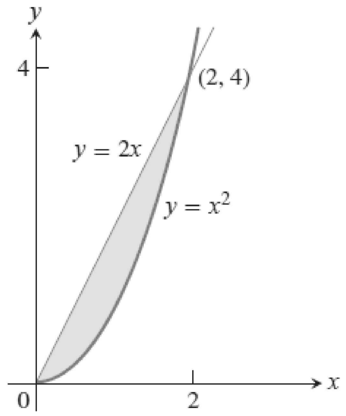
$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$$

and write an equivalent integral with the order of integration reversed.

Solution: the region of integration is given by the inequalities

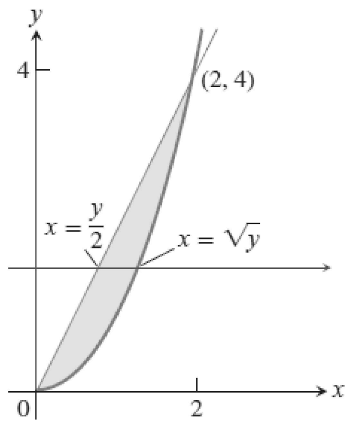
$$x^2 \le y \le 2x \quad \text{and} \quad 0 \le x \le 2$$

It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ Between $x = 0$ and $x = 2$



To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = \frac{y}{2}$ and $x = \sqrt{y}$ .to include all such lines, we let y run from $y = 0$ to $y = 4$. the integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy \quad \rightarrow \textit{the common value of these integral is } 8$$



Properties of double integrals:
It $f(x, y)$ and $g(x, y)$ are continuous, then:

1. *constant multiple*:

$$\int\int_R cf(f(x,y)dA = c\int\int_R f(x,y)dA \quad (any\ number\ c)$$

2. *sum and difference*:

$$\int\int_R (f(x,y) + g(x,y))dA = \int\int_R f(x,y)dA \mp g(x,y)dA$$

3. *Domination*:

a. $\int\int_R f(x,y)dA \geq 0 \quad if \quad f(x,y) \geq 0 \quad on \ R$

b. $\int\int_R f(x,y)dA \geq \int\int_R g(x,y)dA \quad if \quad f(x,y) \geq g(x,y) \quad on \ R$

4. *Additivity*:

$$\int\int_R f(x,y)dA = \int\int_{R_1} f(x,y)dA + \int\int_{R_2} f(x,y)dA$$

If R is the union of two non-overlapping regions R_1 and R_2

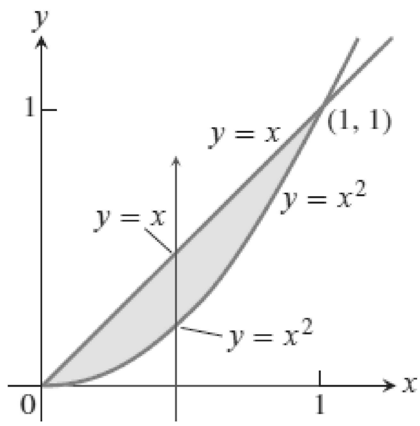
***Area, moment and centers of mass*:**

Areas of bounded regions in the plane:
The area of a closed, bounded plane region R is:

$$A = \iint_R dA$$

Example: Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution: we sketch the region, nothing where the two curves intersect and calculate the area as:



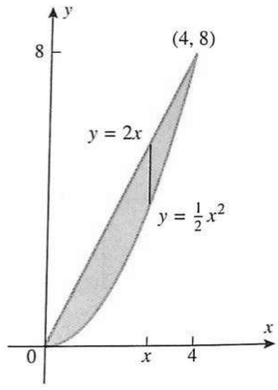
$$A = \int_0^1 \int_{x^2}^{x} dy dx = \int_0^1 [y]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Example: Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$

Solution:



(a)

$$area = \int_0^4 \int_{x^2/2}^{2x} dydx = \int_0^4 [y]_{x^2/2}^{2x}\ dx$$

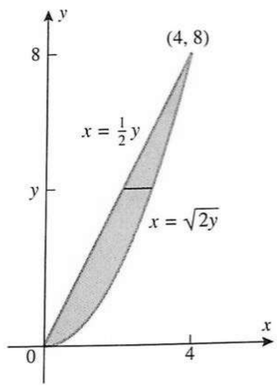
$$= \int_0^4 \left(2x - \frac{x^2}{2}\right) dx$$

$$= \left[\frac{2x^2}{2} - \frac{x^3}{6}\right]_0^4 = \left[x^2 - \frac{x^3}{6}\right]_0^4$$

$$= \left[(4)^2 - \frac{(4)^3}{6}\right] = 16 - \frac{64}{6}$$

$$= \frac{96 - 64}{6} = \frac{32}{6} = \frac{16}{3}$$

or



(b)

$$area = \int_0^8 \int_{y/2}^{\sqrt{2y}} dxdy = \int_0^8 [x]_{y/2}^{\sqrt{2y}} dy$$

$$= \int_0^8 \left(\sqrt{2y} - \frac{y}{2}\right) dy$$

$$= \left[\frac{2\sqrt{2}}{3} y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{2\sqrt{2}}{3}(8)^{3/2} - \frac{(8)^2}{4}$$

$$= \frac{16}{3}$$

Moment and centers of mass for thin flat plates:
Mass and first moment formula for thin plates covering a region R in the xy – plane:

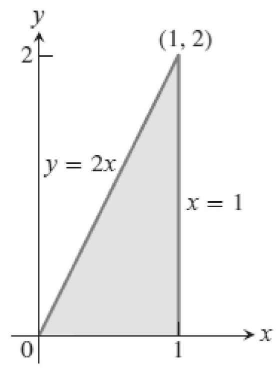
$$\textbf{Mass}:\quad M=\iint_R \delta(x,y)dA \qquad \delta(x,y)\, is\ the\ density\ at\ (x,y)$$

$$\textbf{First moments}:\quad M_x=\iint_R y\delta(x,y)dA \quad ,M_y=\iint_R x\delta(x,y)dA$$

$$\textbf{Center of mass}:\quad \bar{x}=\frac{M_y}{M} \quad , \quad \bar{y}=\frac{M_x}{M}$$

Example: A thin plate covers the triangular region bounded by the x – axis and the line $x=1$ and $y=2x$ in the first quadrant. The plate's density at the point (x,y) is $\delta(x,y)=6x+6y+6$. **find the plate's mass, first moments and center of mass** about the coordinate axes.

Solution: We sketch the plate and put in enough detail to determine the limits of integration for the integrals we have to evaluate:



$$M=\int_0^1\int_0^{2x}\delta(x,y)dydx=\int_0^1\int_0^{2x}(6x+6y+6)dydx$$

$$=\int_0^1\left[6xy+\frac{6y^2}{2}+6y\right]_{y=0}^{y=2x}dx=\int_0^1[6xy+3y^2+6y]_{y=0}^{y=2x}dx$$

$$=\int_0^1[(6x)(2x)+(3)(2x)^2+(6)(2x)]dx$$

$$= \int_0^1 (12x^2 + 12x^2 + 12x)dx$$

$$= \int_0^1 (24x^2 + 12x)dx = \left[\frac{24x^3}{3} + \frac{12x^2}{2}\right]_0^1$$

$$= [8x^3 + 6x^2]_0^1 = [(8)(1) + (6)(1)] = 8 + 6 = 14$$

The first moment about the x – axis is:

$$M_x = \int_0^1 \int_0^{2x} y\delta(x,y)dydx = \int_0^1 \int_0^{2x} y(6x + 6y + 6)dydx$$

$$= \int_0^1 \int_0^{2x} (6xy + 6y^2 + 6y)dydx$$

$$= \int_0^1 \left[\frac{6xy^2}{2} + \frac{6y^3}{3} + \frac{6y^2}{2}\right]_{y=0}^{y=2x} dx = \int_0^1 [3xy^2 + 2y^3 + 3y^2]_{y=0}^{y=2x} dx$$

$$= \int_0^1 [(3x)(2x)^2 + (2)(2x)^3 + (3)(2x)^2]dx$$

$$= \int_0^1 (12x^3 + 16x^3 + 12x^2)dx = \int_0^1 (28x^3 + 12x^2)dx$$

$$= \left[\frac{28x^4}{4} + \frac{12x^3}{3}\right]_0^1 = [7x^4 + 4x^3]_0^1$$

$$= [(7)(1) + (4)(1)] = 7 + 4 = 11$$

A similar calculation gives the moment about the y – axis:

$$M_y = \int_0^1 \int_0^{2x} x\delta(x,y)dydx = \int_0^1 \int_0^{2x} x(6x + 6y + 6)dydx$$

$$= \int_0^1 \int_0^{2x} (6x^2 + 6xy + 6x)dydx$$

$$= \int_0^1 \left[6x^2y + \frac{6xy^2}{2} + 6xy\right]_{y=0}^{y=2x} dx$$

$$= \int_0^1 [6x^2y + 3xy^2 + 6xy]_{y=0}^{y=2x} dx$$

$$= \int_0^1 [(6x^2)(2x) + (3x)(2x)^2 + (6x)(2x)]dx$$

$$= \int_0^1 (12x^3 + 12x^3 + 12x^2)dx$$

$$= \int_0^1 (24x^3 + 12x^2)dx = \left[\frac{24x^4}{4} + \frac{12x^3}{3}\right]_0^1$$

$$= [6x^4 + 4x^3]_0^1 = [(6)(1) + (4)(1)] = 6 + 4 = 10$$

The coordinates of the center of mass are therefore:

$$\bar{x} = \frac{M_y}{M} = \frac{10}{14} = \frac{5}{7}$$

$$\bar{y} = \frac{M_x}{M} = \frac{11}{14}$$

***Moment of inertia*:**

Moment of inertia (second moment):

About the y – axis:

$$I_x = \iint y^2 \delta(x, y) dA$$

About the y – axis:

$$I_y = \iint x^2 \delta(x, y) dA$$

About a line L :

$$I_L = \iint r^2(x, y) \delta(x, y) dA$$

Where $r(x, y) =$ distance from (x, y) to L

About the origin:

$$I_o = \iint (x^2 + y^2) \delta(x, y) dA = I_x + I_y$$

Radii of gyration:

About the x – axis: $R_x = \sqrt{\frac{I_x}{M}}$

About the y – axis: $R_y = \sqrt{\frac{I_y}{M}}$

About the origin: $R_o = \sqrt{\frac{I_o}{M}}$

Example: A thin plate covers the triangle region bounded by the x – axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $\delta(x, y) = 6x + 6y + 6$. **find the moment of inertia and radii of gyration** about the coordinate axes and the origin.

Solution:

moment of inertia about the x – axis is:

$$I_x = \int_0^1 \int_0^{2x} y^2 \delta(x, y) dydx = \int_0^1 \int_0^{2x} y^2 \,(6x + 6y + 6) dydx$$

$$= \int_0^1 \int_0^{2x} (6xy^2 + 6y^3 + 6y^2) dydx$$

$$= \int_0^1 \left[\frac{6xy^3}{3} + \frac{6y^4}{4} + \frac{6y^3}{3}\right]_{y=0}^{y=2x} dx$$

$$= \int_0^1 \left[2xy^3 + \frac{3y^4}{2} + 2y^3\right]_{y=0}^{y=2x} dx$$

$$= \int_0^1 \left[(2x)(2x)^3 + \left(\frac{3}{2}\right)(2x)^4 + (2)(2x)^3\right] dx$$

$$= \int_0^1 \left(16x^4 + \left(\frac{3}{2}\right)(16x^4) + 16x^3\right) dx = \int_0^1 (16x^4 + 24x^4 + 16x^3)\, dx$$

$$= \int_0^1 (40x^4 + 16x^3) dx = \left[\frac{40x^5}{5} + \frac{16x^4}{4}\right]_0^1 = [8x^5 + 4x^4]_0^1$$

$$= [(8)(1) + (4)(1)] = 8 + 4 = 12$$

The moment of inertia about the y – axis is:

$$I_y = \int_0^1 \int_0^{2x} x^2 \delta(x,y)dydx = \int_0^1 \int_0^{2x} x^2\,(6x + 6y + 6)dydx$$

$$= \int_0^1 \int_0^{2x} (6x^3 + 6x^2y + 6x^2)dydx$$

$$= \int_0^1 \left[6x^3y + \frac{6x^2y^2}{2} + 6x^2y\right]_{y=0}^{y=2x} dx$$

$$= \int_0^1 [6x^3y + 3x^2y^2 + 6x^2y]_{y=0}^{y=2x}\, dx$$

$$= \int_0^1 [(6x^3)(2x) + (3x^2)(2x)^2 + (6x^2)(2x)]_{y=0}^{y=2x} dx$$

$$= \int_0^1 (12x^4 + 12x^4 + 12x^3)\, dx$$

$$= \int_0^1 (24x^4 + 12x^3)\, dx$$

$$= \left[\frac{24x^5}{5} + \frac{12x^4}{4}\right]_0^1 = \left[\frac{24x^5}{5} + 3x^4\right]_0^1$$

$$= \left[\left(\frac{24}{5}\right)(1)^5 + (3)(1)^4\right] = \frac{24}{5} + 3 = \frac{24+15}{5} = \frac{39}{5}$$

The moment of inertia about the origin:

$$I_o = \int\int (x^2 + y^2)\delta(x, y)dydx = I_x + I_y$$

$$\therefore \quad I_o = 12 + \frac{39}{5} = \frac{60 + 39}{5} = \frac{99}{5}$$

The three radii of gyration are:

$$R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{12}{14}} = \sqrt{\frac{6}{7}} = 0.926$$

$$R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{39/5}{14}} = \sqrt{\frac{39}{70}} = 0.746$$

$$R_o = \sqrt{\frac{I_o}{M}} = \sqrt{\frac{99/5}{14}} = \sqrt{\frac{99}{70}} = 1.189$$

H.W:

1. Find the **center of mass** and the **moment of inertia** and **radius of gyration** about the y – axis of thin rectangular plate cut from the first quadrant by the lines $x = 6$ and $y = 1$ if $\delta(x, y) = x + y + 1$

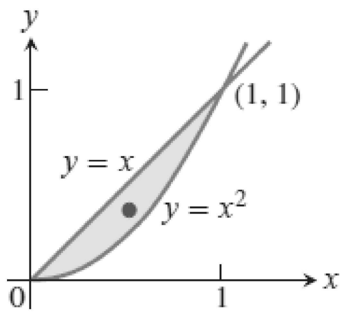
2. Find the **moment of inertia** and **radius of gyration** about the coordinate axes of a thin rectangular plate of constant density δ bounded by the lines $x = 3$ and $y = 3$ in the first quadrant.

Centroid of geometric figures:

When the density of an object is constant, it cancels out of the numerator and denominator of formulas for $\bar{x}$ and $\bar{y}$. as for as $\bar{x}$ and $\bar{y}$ are concerned, δ might as well be 1 thus, when δ is constant, the location of the center of mass becomes a feature of the object's shape and not of the material of which it is made. In such cases, engineers may call the center of mass the **centroid** of the shape. To find a centroid, we set δ equal to 1 and proceed to find $\bar{x}$ and $\bar{y}$ as before, by dividing first moments by masses.

Example: Find the **centroid** of the region in the first quadrant that is bounded above by the line $y = x$ and below by the parabola $y = x^2$

Solution: we sketch the region and include enough detail to determine the limits of integration. We then set δ equal to 1



$$M = \int_0^1 \int_{x^2}^{x} 1 dy dx = \int_0^1 [y]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

$$M_x = \int_0^1 \int_{x^2}^{x} y dy dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=x^2}^{y=x} dx = \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2}\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^5}{10}\right]_0^1 = \frac{1}{6} - \frac{1}{10} =$$

$$M_y = \int_0^1 \int_{x^2}^{x} x dy dx = \int_0^1 [xy]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 [(x)(x) - (x)(x^2)]\, dx = \int_0^1 (x^2 - x^3) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

From these values of M , M_x and M_y , we find:

$$\bar{x} = \frac{M_y}{M} = \frac{\left(\frac{1}{12}\right)}{\left(\frac{1}{6}\right)} = \frac{6}{12} = \frac{1}{2}$$

And

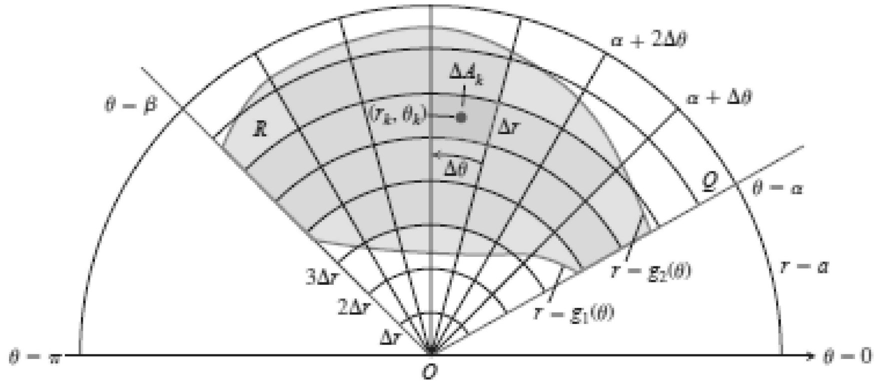
$$\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{1}{15}\right)}{\left(\frac{1}{6}\right)} = \frac{6}{15} = \frac{2}{5}$$

$\therefore$ *the centroid is the point* $\left(\frac{1}{2}, \frac{2}{5}\right)$

Double integrals in polar form

Integrals in polar coordinates:

Suppose that a function $f(r,\theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. suppose also that $0 \le g_1(\theta) \le g_2(\theta) \le a$ for every value of θ between α and β . then R lies in a fan – shaped region Q defined by the inequalities $0 \le r \le a$ and $\alpha \le \theta \le \beta$



We number the polar rectangles that lie inside R (the order does not matter) calling their areas ΔA_1 , ΔA_2 , …… , ΔA_n , we let (r_k, θ_k) be any point in the polar rectangle whose area is ΔA_k . we then form the sum

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k$$

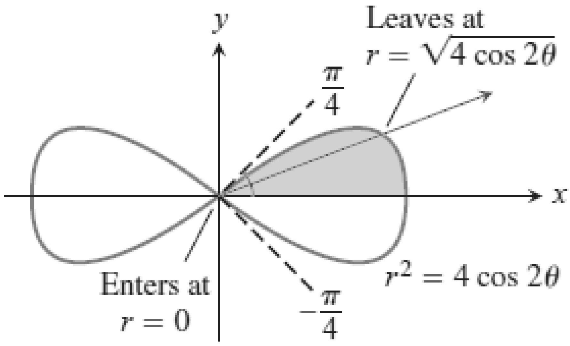
If f is continuous throughout R, this sum will approach a limit as we refine the grid to make Δr and $\Delta\theta$ go to zero. The limit is called the double integral of f over R. in symbols,

$$\lim_{n\to\infty} S_n = \iint_R f(r,\theta)\, dA$$

Area in polar coordinates:
The area of a closed and bounded region R in the polar coordinate plane is:

$$A = \int\int_R r dr d\theta$$

Example: Find the **area** enclosed by $r^2 = 4\cos 2\theta$



Solution:

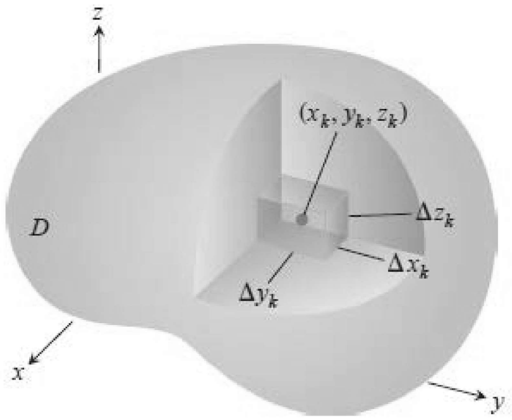
$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4cos2\theta}} r dr d\theta = 4\int_0^{\pi/4} \left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4cos2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2cos2\theta d\theta = [4sin2\theta]_0^{\pi/4} = 4$$

Triple integrals in rectangular coordinates:

triple integrals:

If $f(x, y, z)$ is a function defined on a closed bounded region *D* in space, such as the region occupied by a solid ball or a lump of clay, then the integral of *F* over *D* may be defined in the following way. We partition a rectangular boxlike region containing *D* into rectangular cells by planes parallel to the coordinate axis



We number the cells that lie inside *D* from 1 to n some order, the *kth* cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. we choose a point (x_k, y_k, z_k) in each cell and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k$$

The ***triple integral of F over D*** :

$$\lim_{n\to\infty} S_n = \int\int_D\int F(x, y, z)dV$$

or

$$\lim_{\|P\|\to 0} S_n = \int\int_D\int F(x, y, z)dxdydz$$

Volume:
The volume of a closed, bounded region *D* in space is:

$$V = \iiint_D dV$$

Example: Evaluate the integral

$$\int_0^1 \int_x^1 \int_0^{y-x} dzdydx$$

Solution:

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dzdydx = \int_0^1 \int_x^1 [z]_{z=0}^{z=y-x} dydx$$

$$= \int_0^1 \int_x^1 (y-x)\, dydx = \int_0^1 \left[\frac{y^2}{2} - xy\right]_{y=x}^{y=1} dx$$

$$= \int_0^1 \left[\left(\frac{1}{2} - x\right) - \left(\frac{x^2}{2} - x^2\right)\right] dx$$

$$= \int_0^1 \left[\frac{1}{2} - x - \frac{x^2}{2} + x^2\right] dx = \int_0^1 \left[\frac{1}{2} - x + \frac{x^2}{2}\right] dx$$

$$= \left[\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}$$

Example: Evaluate the integral

$$\int_0^1 \int_0^{1-y} \int_0^2 dxdzdy$$

Solution:

$$V = \int_0^1 \int_0^{1-y} \int_0^2 dxdzdy = \int_0^1 \int_0^{1-y} [x]_{x=0}^{x=2} \, dzdy$$

$$= \int_0^1 \int_0^{1-y} 2 \, dzdx = \int_0^1 [2z]_{z=0}^{z=1-y} \, dy$$

$$= \int_0^1 2(1-y)dy = \int_0^1 (2-2y)dy$$

$$= \left[2y - \frac{2y^2}{2}\right]_0^1 = [2y - y^2]_0^1$$

$$= 2 - 1 = 1$$

Example: Evaluate the integral

$$\int_0^1 \int_0^2 \int_0^{1-z} dydxdz$$

Solution:

$$V = \int_0^1 \int_0^2 \int_0^{1-z} dydxdz = \int_0^1 \int_0^2 [y]_0^{1-z}\, dxdz$$

$$= \int_0^1 \int_0^2 (1-z)dxdz = \int_0^1 [x - zx]_{x=0}^{x=2}\ dz$$

$$= \int_0^1 (2-2z)dz = \left[2z - \frac{2z^2}{2}\right]_0^1 = [2z - z^2]_0^1$$

$$= 2 - 1 = 1$$

***Example*: Evaluate the integral**

$$\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dzdydx$$

***Solution*:**

$$V=\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dzdydx=\int_0^1\int_0^1\left[x^2z+y^2z+\frac{z^3}{3}\right]_{z=0}^{z=1}dydx$$

$$=\int_0^1\int_0^1\left(x^2+y^2+\frac{1}{3}\right)dydx$$

$$=\int_0^1\left[x^2y+\frac{y^3}{3}+\frac{y}{3}\right]_{y=0}^{y=1}dx=\int_0^1\left(x^2+\frac{1}{3}+\frac{1}{3}\right)dx$$

$$=\int_0^1\left(x^2+\frac{2}{3}\right)dx=\left[\frac{x^3}{3}+\frac{2}{3}x\right]_0^1=\frac{1}{3}+\frac{2}{3}=\frac{3}{3}=1$$

***H.W*: Evaluate the integral:**

1. $$\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$$

2. $$\int_0^1\int_0^{1-x}\int_{x+z}^{1} dydzdx$$

3. $$\int_{-1}^1\int_{-1}^1\int_{-1}^1 (x+y+z)\,dydxdz$$

Masses and moments in three dimensions:

Mass and moment formulas for solid objects in space:

Mass:

$$M = \iiint_D \delta dV \qquad (\delta = \delta(x,y,z) = density)$$

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\delta dV$$

$$M_{xz} = \iiint_D y\delta dV$$

$$M_{xy} = \iiint_D z\delta dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M} \quad , \bar{y} = \frac{M_{xz}}{M} \quad , \bar{z} = \frac{M_{xy}}{M}$$

Moments of inertia (second moments) about the coordinate axes:

$$I_x = \int\int\int (y^2 + z^2)\delta dV$$

$$I_y = \int\int\int (x^2 + z^2)\delta dV$$

$$I_z = \int\int\int (x^2 + y^2)\delta dV$$

Moments of inertia about a line L:

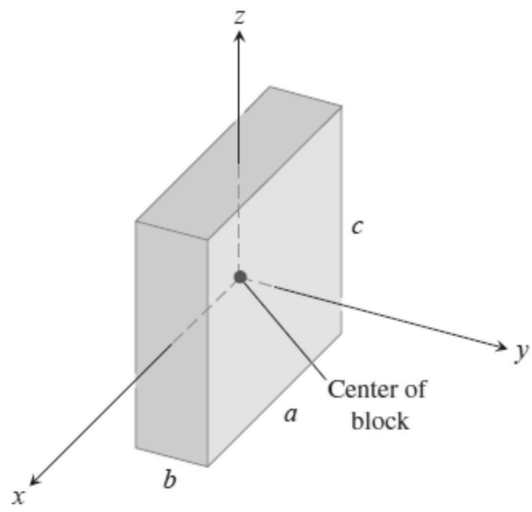
$$I_L = \int\int\int r^2 \delta dV$$

($r(x, y, z) =$ Distance from the point (x, y, z) to the line L)

Radius of gyration about a line L:

$$R_L = \sqrt{\frac{I_L}{M}}$$

Example: Find I_x , I_y , I_z for the rectangular solid of constant density δ shown in the figure



Solution: the rectangular solid consists of eight symmetric pieces, one in each octant. We can evaluate the integral on one of these pieces and then multiply by 8 to get the total value

$$I_x = 8\int_0^{c/2}\int_0^{b/2}\int_0^{a/2} (y^2 + z^2)\,\delta dxdydz$$

$$= 8\delta\int_0^{c/2}\int_0^{b/2} [y^2x + z^2x]_{x=0}^{x=a/2}\,dydz$$

$$= 8\delta\int_0^{c/2}\int_0^{b/2} \left[(y^2)\left(\frac{a}{2}\right) + (z^2)\left(\frac{a}{2}\right)\right] dydz$$

$$= \frac{8a\delta}{2}\int_0^{c/2}\int_0^{b/2} [y^2 + z^2]dydz = 4a\delta\int_0^{c/2}\int_0^{b/2} [y^2 + z^2]dydz$$

$$= 4a\delta\int_0^{c/2} \left[\frac{y^3}{3} + z^2y\right]_{y=0}^{y=b/2} dz$$

$$= 4a\delta \int_0^{c/2} \left[\frac{(b/2)^3}{3} + (z^2)\left(\frac{b}{2}\right)\right] dz$$

$$= 4a\delta \int_0^{c/2} \left(\frac{b^3}{24} + \frac{z^2 b}{2}\right) dz$$

$$= 4a\delta \left[\frac{b^3 z}{24} + \frac{z^3 b}{6}\right]_0^{c/2}$$

$$= 4a\delta \left[\frac{(b^3)(c/2)}{24} + \frac{(c/2)^3 b}{6}\right]$$

$$= 4a\delta \left(\frac{b^3 c}{48} + \frac{c^3 b}{48}\right) = \frac{4abc\delta}{48}(b^2 + c^2)$$

∴

$$I_x = \frac{abc\delta}{12}(b^2 + c^2) = \frac{M}{12}(b^2 + c^2)$$

$$I_y = 8 \int_0^{c/2} \int_0^{b/2} \int_0^{a/2} (x^2 + z^2)\, \delta dx dy dz$$

$$= 8\delta \int_0^{c/2} \int_0^{b/2} \left[\frac{x^3}{3} + z^2 x\right]_{x=0}^{x=a/2} dy dz$$

$$= 8\delta \int_0^{c/2} \int_0^{b/2} \left[\frac{(a/2)^3}{3} + (z^2)\left(\frac{a}{2}\right)\right] dy dz$$

$$= 8\delta \int_0^{c/2} \int_0^{b/2} \left(\frac{a^3}{24} + \frac{z^2 a}{2} \right) dydz$$

$$= \frac{8a\delta}{2} \int_0^{c/2} \int_0^{b/2} \left(\frac{a^2}{12} + z^2 \right) dydz = 4a\delta \int_0^{c/2} \int_0^{b/2} \left(\frac{a^2}{12} + z^2 \right) dydz$$

$$= 4a\delta \int_0^{c/2} \left[\frac{a^2 y}{12} + z^2 y \right]_{y=0}^{y=b/2} dz$$

$$= 4a\delta \int_0^{c/2} \left[\frac{(a^2)(b/2)}{12} + (z^2)\left(\frac{b}{2}\right) \right] dz$$

$$= 4a\delta \int_0^{c/2} \left[\frac{a^2 b}{24} + \frac{z^2 b}{2} \right] dz$$

$$= \frac{4ab\delta}{2} \int_0^{c/2} \left[\frac{a^2}{12} + z^2 \right] dz = 2ab\delta \left[\frac{a^2 z}{12} + \frac{z^3}{3} \right]_0^{c/2}$$

$$= 2ab\delta \left[\frac{(a^2)\left(\frac{c}{2}\right)}{12} + \frac{\left(\frac{c}{2}\right)^3}{3} \right]$$

$$= 2ab\delta \left[\frac{a^2 c}{24} + \frac{c^3}{24} \right]$$

$$= \frac{2abc\delta}{24}(a^2 + c^2) = \frac{abc\delta}{12}(a^2 + c^2)$$

$\therefore$

$$I_y = \frac{M}{12}(a^2 + c^2)$$

$$I_z = 8\int_0^{c/2}\int_0^{b/2}\int_0^{a/2} (x^2+y^2)\,\delta dxdydz$$

$$= 8\delta\int_0^{c/2}\int_0^{b/2} \left[\frac{x^3}{3}+y^2x\right]_{x=0}^{x=a/2} dydz$$

$$= 8\delta\int_0^{c/2}\int_0^{b/2} \left[\frac{(a/2)^3}{3}+(y^2)\left(\frac{a}{2}\right)\right] dydz$$

$$= 8\delta\int_0^{c/2}\int_0^{b/2} \left(\frac{a^3}{24}+\frac{y^2a}{2}\right) dydz = \frac{8a\delta}{2}\int_0^{c/2}\int_0^{b/2} \left(\frac{a^2}{12}+y^2\right) dydz$$

$$= 4a\delta\int_0^{c/2} \left[\frac{a^2y}{12}+\frac{y^3}{3}\right]_{y=0}^{y=b/2} dz$$

$$= 4a\delta\int_0^{c/2} \left[\frac{(a^2)(b/2)}{12}+\frac{(b/2)^3}{3}\right] dz = 4a\delta\int_0^{c/2} \left(\frac{a^2b}{24}+\frac{b^3}{24}\right) dz$$

$$= \frac{4ab\delta}{24}\int_0^{c/2} (a^2+b^2)dz$$

$$= \frac{ab\delta}{6}[a^2z+b^2z]_0^{c/2} = \frac{ab\delta}{6}\left[(a^2)\left(\frac{c}{2}\right)+(b^2)\left(\frac{c}{2}\right)\right]$$

$$= \frac{ab\delta}{6}\left(\frac{a^2c}{2}+\frac{b^2c}{2}\right) = \frac{abc\delta}{12}(a^2+b^2)$$

$\therefore$

$$I_z = \frac{M}{12}(a^2+b^2)$$

***H.W*:** Find the **center of mass** of a solid of constant density δ bounded below by the disk R: $x^2 + y^2 \le 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$